16.6 Propagation of Electromagnetic Waves Between Parallel Conducting Plates

A structure consisting of highly conducting walls in the form of a long open pipe is used to guide electromagnetic waves in the microwave region (3–300 GHz) from one place to other. Such a structure is called a *waveguide*. It may be studied as a boundary value problem with multiple boundaries. As a preliminary to the study of such waveguides we consider here a simpler boundary value problem-namely, the propagation of electromagnetic waves in the region between two parallel, perfectly conducting plates.

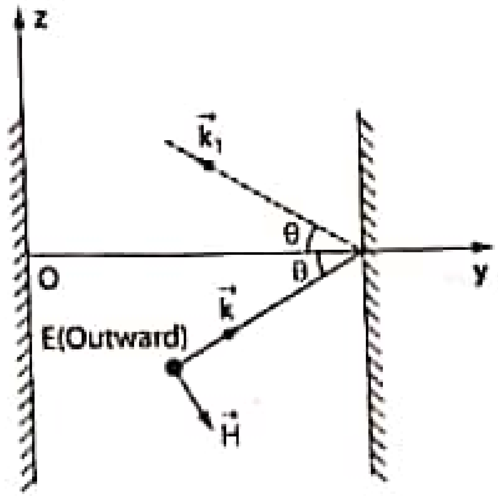

Fig 16.6-1: Wave propagation between two parallel conducting plates.

Suppose we consider two perfectly conducting plates located at $y = 0$ and $y = a$ as shown in Fig 16.6-1. The space between the plates is assumed to be free of charges, currents and have permittivity and permeability equal to those of vacuum. Let an electromagnetic wave of frequency ω with wave vector $\vec{k}$ in the yz-plane be incident on the $y = a$ plate with an angle of incidence θ. Note that in Fig 16.6-1 the x- and z-directions are physically indistinguishable and hence no generality is lost by considering only waves with wave vectors in the yz-plane. The wave incident on $y = a$ plate will be reflected between the plates by the process of multiple reflections and in effect it will propagate in a direction parallel to the plates. For such waves there are two possible polarizations. The electric field $\vec{E}$ may be parallel to x-axis, i.e., at right angles to the direction o[f] propagation (s-polarization). This is called *transverse electric (TE) wave*. If the $\vec{H}$-fiel[d] is along x-axis, i.e., perpendicular to the direction of propagation (p-polarization) the corresponding wave is called *transverse magnetic (TM) wave*. Here we shall conside[r] only the TE-waves. The electric field in the region between the plates is given by th[e] sum of incident and reflected waves. Thus, the electric field in the region between th[e] plates for TE waves may be described by

$$\vec{E} = \hat{i}E_0' e^{j(\vec{k}\cdot\vec{r}-\omega t)} + \hat{i}E_{10}' e^{j(\vec{k}_1\cdot\vec{r}-\omega t)}. \qquad (16.6\text{-}$$

Power flow associated with TM modes

Time average power transmitted through the waveguide in TM modes is given by

$$P = \int_0^a \int_0^b \frac{1}{2} Re \left(\vec{E} \times \vec{H}^* \right) \cdot \hat{k} dx dy.$$

As before,

$$P = \frac{1}{2} Re \int_0^a \int_0^b \left(E_{ox} H_{oy}^* - E_{oy} H_{ox}^* \right) dx dy.$$

Using the field components given by (16.7-30) we can get

$$P = \frac{ab}{8} \cdot \frac{\omega \epsilon_0 k_g}{h^2} \cdot E_0^2.$$

Impossibility of TEM mode in a hollow waveguide

Suppose a TEM mode, for which there is no axial component of either $\vec{E}$ or $\vec{H}$, is assumed to exist in a hollow waveguide. In presence of nonmagnetic material $\vec{\nabla} \cdot \vec{H} = 0$ So if a TEM mode exists lines of $\vec{H}$ will be closed loops lying entirely in transverse plane. Now, from Maxwell's equation,

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

we can write

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S},$$

i.e., the magnetomotive force around these closed loops must be equal to axial current through the loops. For a hollow waveguide having no inner conductor conduction current is zero. Further, any axial displacement current requires an axial component of $\vec{E}$, something not present in TEM mode. Thus, we conclude that TEM mode cannot exist in a hollow waveguide.

But such a mode can exist in coaxial cable and in parallel line transmission system.

16.8 Cavity Resonator

A cavity resonator is an enclosure which completely surrounds electromagnetic fields by means of highly conducting walls. Such a cavity can sustain stationary waves of only certain discrete frequencies, called its *eigen frequencies*. It has properties similar to ordinary LC resonant circuits. In microwave region, where ordinary LC circuits cannot work cavity resonators are invariably used as resonant circuits.

Now the condition $E_{oz} = 0$ at $x = a$ requires that

$$\sin k_x a = 0 \quad \text{or} \quad k_x = \frac{m\pi}{a}; \quad m = 0, 1, 2, \ldots$$

Similarly, the condition $E_{oz} = 0$ at $y = b$ implies that

$$\sin k_y b = 0 \quad \text{or} \quad k_y = \frac{n\pi}{b}; \quad n = 0, 1, 2, \ldots$$

Thus, finally we get

$$E_{oz}(x, y) = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{16.7-30a}$$

Now as E_{oz} is known other field components can be easily calculated in terms of it.

$$E_{ox} = \frac{jk_g}{h^2}\frac{\partial E_{oz}}{\partial x} = \frac{jk_g}{h^2} \cdot E_0 \left(\frac{m\pi}{a}\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$E_{oy} = \frac{jk_g}{h^2}\frac{\partial E_{oz}}{\partial y} = \frac{jk_g}{h^2} E_0 \left(\frac{m\pi}{b}\right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$H_{ox} = -\frac{j\omega\epsilon_0}{h^2}\frac{\partial E_{oz}}{\partial y} = -\frac{j\omega\epsilon_0}{h^2} E_0 \left(\frac{n\pi}{b}\right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$H_{oy} = \frac{j\omega\epsilon_0}{h^2} \cdot \frac{\partial E_{oz}}{\partial x} = \frac{j\omega\epsilon_0}{h^2} E_0 \left(\frac{m\pi}{a}\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{16.7-30b}$$

As before we can calculate the cut off wavelength, which is the same as given by (16.7-23). In TM mode all the field components vanish for both m and n zero or any one of them zero. Thus, $(TM)_{00}$, $(TM)_{10}$ or $(TM)_{01}$ modes are not possible. For TM mode the longest wavelength available for transmission though the guide is

$$(\lambda_c)_{11} = \frac{2}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}},$$

which is definitely smaller than that for TE mode, where the longest wavelength is

$$(\lambda_c)_{10} = \frac{2}{\sqrt{\left(\frac{1}{a}\right)^2}} \quad \text{(assuming } a > b\text{)}$$

This result shows that the frequency band available for transmission through the waveguide is smaller in TM mode than that in TE mode. In this respect, TE mode is superior to TM mode of propagation.

Power flow associated with TE waves

The time average power flowing through the wavelength along z-axis is given by

$$
\begin{aligned}
P &= \int_0^a \int_0^b \frac{1}{2} Re\left(\vec{E} \times \vec{H}^*\right) \cdot \hat{k}\, dx dy \\
&= \frac{1}{2} Re \int_0^a \int_0^b \left(E_x H_y^* - E_y H_x^*\right) dx dy \\
&= \frac{1}{2} Re \int_0^a \int_0^b \left(E_{ox} H_{oy}^* - E_{oy} H_{ox}^*\right) dx dy \\
&= \frac{1}{2} \frac{\omega \mu_0 k_g}{h^4} H_0^2 \int_0^a \int_0^b \left[\left(\frac{n\pi}{b}\right)^2 \cdot \cos^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \right. \\
&\qquad \left. + \left(\frac{m\pi}{a}\right)^2 \sin^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b}\right] dx dy \\
&= \frac{H_0^2 \omega \mu_0 k_g}{2h^4} \cdot \frac{a}{2} \cdot \frac{b}{2} \cdot \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \\
&= \frac{ab}{8} \cdot \frac{\omega \mu_0 k_g}{h^2} \cdot H_0^2 \quad \left[\because h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \qquad (16.7\text{-}29)
\end{aligned}
$$

For either $m = 0$ or $n = 0$, we have

$$P = \frac{ab}{4} \cdot \frac{\omega \mu_0 k_g}{h^2} \cdot H_0^2.$$

Thus, for a particular waveguide power transmission will be greater for larger H_0. A waveguide, however, cannot transmit any arbitrarily large power. The maximum value is limited by the puncture of the dielectric filling the waveguide.

TM modes

In this case, the waveguide contains no z-component of magnetic field, i.e., $H_{oz} = 0$ but $E_{oz} \neq 0$. Here we start from the equation satisfied by E_z and proceeding as before we can write for the most general solution of E_{oz} as

$$E_{oz}(x, y) = \left(A_1 \cos k_x x + A_2 \sin k_x x\right)\left(A_3 \cos k_y y + A_4 \sin k_y y\right).$$

To find the constants A_i we apply the boundary conditions $E_{oz} = 0$ at $x = 0, x = a$, $y = 0, y = b$.

$E_{oz} = 0$ at $x = 0$ requires that $A_1 = 0$.

$E_{oz} = 0$ at $y = 0$ requires that $A_3 = 0$.

Thus,

$$E_{oz} = E_0 \sin k_x x \sin k_y y,$$

where $E_0 = A_2 A_4$ is determined by introduced field.

This is called *cut-off frequency*. All frequency components greater than ω_c can propagate through the guide and below it propagation is not possible. The *cut-off wavelength* is given by

$$\lambda_c = \frac{c}{\omega_c/2\pi} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}. \qquad (16.7\text{-}23)$$

Propagation is possible for wavelength $\lambda < \lambda_c$.

Now that the values of ω_c or λ_c depend on the transverse dimensions of the waveguide and also on the mode (m, n) of propagation. TE mode for arbitrary values of m and n is denoted by $(TE)_{mn}$. A mode for which λ_c is highest is called a *dominant mode*. Note that $(TE)_{00}$ mode is not possible because for $m = n = 0$, $H_{oz} = H_0$ and all other field components vanish; further, the condition $\vec{\nabla} \cdot \vec{H} = 0$ requires that $ik_g H_z = 0$ or, $H_z = 0$.

From (16.7-21) and (16.7-22) we can write

$$k_g^2 = \frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2}. \qquad (16.7\text{-}24)$$

The quantity $\frac{2\pi}{k_g} = \lambda_g$ is called *guide wavelength*. $\lambda_0 = \frac{c}{\omega/2\pi}$ is the free space wavelength. So we can write from (16.7-24)

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2} \quad \text{or} \quad \lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}} \qquad (16.7\text{-}25)$$

Thus, guide wavelength is greater than corresponding free space wavelength. The phase velocity of the wave along z-direction is

$$v_p = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2}}} = \frac{c}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}. \qquad (16.7\text{-}26)$$

Obviously phase velocity of the wave is greater than the velocity of light in free space. This does not violate the principle of relativity because the group velocity with which energy propagates along the guide is always smaller than c. Using (16.7-24) we can show that the group velocity

$$v_g = \frac{d\omega}{dk_g} = c\sqrt{1 - (\lambda_0/\lambda_c)^2}. \qquad (16.7\text{-}27)$$

It is interesting to note that

$$v_p v_g = c^2. \qquad (16.7\text{-}28)$$

requires that

$$\sin k_x a = 0 \quad \text{or} \quad k_x = \frac{m\pi}{a}; \quad m = 0, 1, 2, 3 \cdots$$

Similarly, the condition

$$E_{ox} \propto \frac{\partial H_{oz}}{\partial y} = 0 \quad \text{at } y = b$$

requires that

$$\sin k_y b = 0 \quad \text{or} \quad k_y = \frac{n\pi}{b}; \quad n = 0, 1, 2, 3, \ldots$$

Thus, finally we get

$$H_{oz}(x, y) = H_0 \cos \frac{m\pi x}{a} \cdot \cos \frac{n\pi y}{b} \tag{16.7-20a}$$

Since H_{oz} is now known other field components can be easily calculated by using the relations (16.7-12)–(16.7-16).

$$E_{ox} = -\frac{j\omega\mu_0}{h^2} H_0 \left(\frac{n\pi}{b}\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$E_{oy} = \frac{j\omega\mu_0}{h^2} H_0 \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$H_{ox} = -\frac{jk_g}{h^2} H_0 \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$H_{oy} = -\frac{jk_g}{h^2} H_0 \left(\frac{n\pi}{b}\right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{16.7-20b}$$

Now

$$h^2 = k_x^2 + k_y^2 \quad \text{or} \quad \frac{\omega^2}{c^2} - k_g^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\text{or} \quad k_g^2 = \frac{\omega^2}{c^2} - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]. \tag{16.7-21}$$

If

$$\omega^2 < c^2 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

then k_g is imaginary. As the z-dependence of the field quantities is assumed to be of the form $e^{jk_g z}$, the fields will be exponentially damped in z, instead of propagating. Thus, for the waves to propagate along the guide the frequency must be greater than some critical value given by

$$\omega_c = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}. \tag{16.7-22}$$

Since the plates are assumed to be perfectly conducting the tangential components of $\vec{E}$-field must vanish at the boundary $y = 0$. For this condition to be true for all values of z and t we must have

$$E_0' = -E_{10}'. \tag{16.6-2}$$

This indicates that the reflected wave will have the same amplitude as the incident wave but with a phase change of π. Therefore,

$$\vec{E} = \hat{i}E_0'\left[e^{j(\vec{k}\cdot\vec{r}-\omega t)} - e^{j(\vec{k}_1\cdot\vec{r}-\omega t)}\right]. \tag{16.6-3}$$

Now we have $\vec{k}\cdot\vec{r} = k\cos\theta\cdot y + k\sin\theta\cdot z$ and $\vec{k}_1\cdot\vec{r} = -k\cos\theta\cdot y + k\sin\theta\cdot z$. Thus,

$$\begin{aligned}\vec{E} &= \hat{i}E_0'\left[e^{jky\cos\theta} - e^{-jky\cos\theta}\right]\cdot e^{j(kz\sin\theta-\omega t)}\\ &= \hat{i}E_0\sin(ky\cos\theta)e^{j(kz\sin\theta-\omega t)},\end{aligned} \tag{16.6-4}$$

where $E_0 = 2jE_0'$. We now apply the boundary condition that the $\vec{E}$-field must vanish at $y = a$ for all values z and t. This requires that

$$\sin(ka\cos\theta) = 0 \quad \text{or} \quad ka\cos\theta = n\pi, \tag{16.6-5}$$

where n is a positive integer. Thus, for a given k and a, there are a number of discrete directions or modes possible for the propagation of waves between the plates.

For convenience let us rewrite Eq. (16.6-4) as

$$\vec{E} = \hat{i}E_0\sin(k_c y)e^{j(k_g z-\omega t)}, \tag{16.6-6}$$

where

$$k_c = k\cos\theta \quad \text{and} \quad k_g = k\sin\theta. \tag{16.6-7}$$

Equation (16.6-6) may how be considered to represent a disturbance propagating along z-axis with an effective wavelength

$$\lambda_g = \frac{2\pi}{k_g} = \frac{2\pi}{k\sin\theta} = \frac{\lambda_0}{\sin\theta}, \tag{16.6-8}$$

where $\lambda_0 = 2\pi/k$ is the free space wavelength. The disturbance also varies sinusoidally along y-axis with an effective wavelength

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{k\cos\theta} = \frac{\lambda_0}{\cos\theta}. \tag{16.6-9}$$

From Eqs. (16.6-8) and (16.6-9) it follows immediately that

$$\frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} = \frac{1}{\lambda_0^2}. \tag{16.6-10}$$

Eliminating H_{oy} from Eqs. (16.7-6) and (16.7-10) we get

$$E_{ox} = \frac{j}{h^2}\left[k_g\frac{\partial E_{oz}}{\partial x} + \omega\mu_0\frac{\partial H_{oz}}{\partial y}\right], \tag{16.7-12}$$

where

$$h^2 = \frac{\omega^2}{c^2} - k_g^2. \tag{16.7-13}$$

Similarly considering other equations appropriately we can express E_{oy}, H_{ox} and H_{oy} in terms of E_{oz} and H_{oz} only.

$$E_{oy} = \frac{j}{h^2}\left[k_g\frac{\partial E_{oz}}{\partial y} - \omega\mu_0\frac{\partial H_{oz}}{\partial x}\right] \tag{16.7-14}$$

$$H_{ox} = \frac{j}{h^2}\left[k_g\frac{\partial H_{oz}}{\partial x} - \omega\epsilon_0\frac{\partial E_{oz}}{\partial y}\right] \tag{16.7-15}$$

$$H_{oy} = \frac{j}{h^2}\left[k_g\frac{\partial H_{oz}}{\partial y} + \omega\epsilon_0\frac{\partial E_{oz}}{\partial x}\right] \tag{16.7-16}$$

Thus, all the transverse components are being completely determined by the longitudinal components and hence the problem of determining fields inside the waveguide reduces to that of determining longitudinal field components only. In general both E_{oz} and H_{oz} are present. However, we shall consider two separate situations:

(*i*) If $E_{oz} = 0$ then entire field is determined by H_{oz}. For this corresponding waves are called *H-waves*. Since here only transverse components of electric field exist, it is also called *transverse electric or TE waves*.

(*ii*) If $H_{oz} = 0$ then entire field is determined by E_{oz}. For this corresponding waves are called *E-waves*. Since here only transverse components of magnetic field exist, it is also called *transverse magnetic or TM waves*.

As Maxwell's equations are linear, by studying TE and TM waves separately we can then superpose them to get the actual picture. There may be a third choice, where both E_{oz} and H_{oz} are zero. Corresponding waves are called *transverse electromagnetic waves or TEM waves*. But Eqs. (16.7-6) to (16.7-11) shows that in this case, the entire field inside the waveguide vanishes identically. It indicates that *pure TEM waves cannot exist in a rectangular waveguide*.

TE modes

In this case, $E_{oz} = 0$ and all other field components may be determined in terms of H_{oz}. From (16.7-2) we find that the equation satisfied by the z-component of $\vec{H}$ is

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) H_z = 0.$$

Taking the z-dependence of H_z of the form $e^{jk_z z}$ this equation can be transformed to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2\right) H_{oz} = 0 \tag{16.7-17}$$

To solve this equation by the method of separation of variables we assume

$$H_{oz}(x, y) = F_1(x) \cdot F_2(y).$$

Substituting it in Eq. (16.7-17) and dividing throughout by $F_1 F_2$ we get

$$\frac{1}{F_1}\frac{d^2F_1}{dx^2} + \frac{1}{F_2}\frac{d^2F_2}{dy^2} = -h^2.$$

Let

$$\frac{1}{F_1}\frac{d^2F_1}{dx^2} = -k_x^2 \quad \text{and} \quad \frac{1}{F_2}\frac{d^2F_2}{dy^2} = -k_y^2,$$

where $k_x^2 + k_y^2 = h^2$.

So the most general solution for $H_{oz}(x, y)$ is

$$H_{oz}(x, y) = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y), \tag{16.7-18}$$

where A_i are arbitrary constants to be evaluated from the given boundary conditions.
The boundary conditions are

$$E_y = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a,$$
$$E_x = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b$$

For TE mode $E_{oz} = 0$ and, therefore, from Eq. (16.7-14),

$$E_{oy} \propto \frac{\partial H_{oz}}{\partial x} = 0 \text{ at } x = 0.$$

This condition requires that $A_2 = 0$ in Eq. (16.7-18). From Eq. (16.7-12),

$$E_{ox} \propto \frac{\partial H_{oz}}{\partial y} = 0 \text{ at } y = 0.$$

This condition requires that $A_4 = 0$ in Eq. (16.7-18). Thus,

$$H_{oz}(x, y) = H_0 \cos k_x x \cdot \cos k_y y, \tag{16.7-19}$$

where $H_0 = A_1 A_3$ and is determined by the introduced field.

Now the condition $$E_{oy} \propto \frac{\partial H_{oz}}{\partial x} = 0 \quad \text{at } x = a$$

We consider a rectangular waveguide of sides a and b as shown in Fig 16.7-1. Let the wave is propagating along z-axis. If the waves to be propagated are monochromatic with time dependence $e^{-j\omega t}$ then Eq. (16.7-1) takes the form

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{X} = 0. \tag{16.7-2}$$

and Maxwell's equations

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{and} \quad \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

become

$$\vec{\nabla} \times \vec{H} = -j\omega\epsilon_0 \vec{E} \tag{16.7-3}$$

$$\text{and} \quad \vec{\nabla} \times \vec{E} = j\omega\mu_0 \vec{H}. \tag{16.7-4}$$

We seek solutions to Eq. (16.7-2) in the form

$$\vec{X}(x, y, z, t) = \vec{X}_0(x, y) e^{j(k_g z - \omega t)}. \tag{16.7-5}$$

Equation (16.7-3) may be written in component forms as

$$\begin{aligned}
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= -j\omega\epsilon_0 E_x; \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= -j\omega\epsilon_0 E_y; \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= -j\omega\epsilon_0 E_z.
\end{aligned}$$

Using solutions (16.7-5) we may write

$$\frac{\partial H_{oz}}{\partial y} - jk_g H_{oy} = -j\omega\epsilon_0 E_{ox}; \tag{16.7-6}$$

$$jk_g H_{ox} - \frac{\partial H_{oz}}{\partial x} = -j\omega\epsilon_0 E_{oy}; \tag{16.7-7}$$

$$\frac{\partial H_{oy}}{\partial x} - \frac{\partial H_{ox}}{\partial y} = -j\omega\epsilon_0 E_{oz}. \tag{16.7-8}$$

Similarly from Eq. (16.7-4) we may get

$$\frac{\partial E_{oz}}{\partial y} - jk_g E_{oy} = j\omega\mu_0 H_{ox}; \tag{16.7-9}$$

$$jk_g E_{ox} - \frac{\partial E_{oz}}{\partial x} = j\omega\mu_0 H_{oy}; \tag{16.7-10}$$

$$\frac{\partial E_{oy}}{\partial x} - \frac{\partial E_{ox}}{\partial y} = j\omega\mu_0 H_{oz}. \tag{16.7-11}$$

Differentiating with respect to k_g we get

$$2k_g = \frac{2\omega}{c^2}\frac{d\omega}{dk_g} \quad \text{or} \quad \frac{d\omega}{dk_g} = \frac{c^2 k_g}{\omega} = \frac{c^2 \cdot k \sin\theta}{ck} = c\sin\theta. \tag{16.6-15}$$

Therefore, group velocity $v_g = c\sin\theta$, which is smaller than the speed of light in vacuum. Moreover, it is interesting to note that

$$v_p \cdot v_g = \frac{c}{\sin\theta} \cdot c\sin\theta = c^2, \tag{16.6-16}$$

which is generally true for propagation in a waveguide.

16.7 Rectangular Waveguide

Waveguides may be in the form of hollow pipes with highly conducting walls (usually of copper) having arbitrary shape but of uniform cross-section. For simplicity we shall consider the special case of a rectangular waveguide with air as dielectric inside. Basically it is a hollow pipe having uniform rectangular cross-section. A wire antenna is often used to launch electromagnetic waves inside the waveguide. For effective transmission through the waveguide the transverse dimensions of the guide must be comparable to the wavelength of the electromagnetic waves to be propagated. The waves launched into the guide propagate by a process of multiple internal reflections. The resulting field configurations can be obtained by solving Maxwell's equations with proper boundary conditions. We shall assume the walls to be made of perfect conductors. So the tangential components of $\vec{E}$-fields and normal components of $\vec{H}$-field must vanish at the boundary.

Assuming the space inside the waveguide to be air or vacuum it is easy to show by using Maxwell's equations that $\vec{E}$ and $\vec{H}$ satisfy the homogenous wave equation [see Section 15.5]

$$\nabla^2 \vec{X} - \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = 0; \quad \left(\vec{X} \equiv \vec{E} \text{ or } \vec{H}\right). \tag{16.7-1}$$

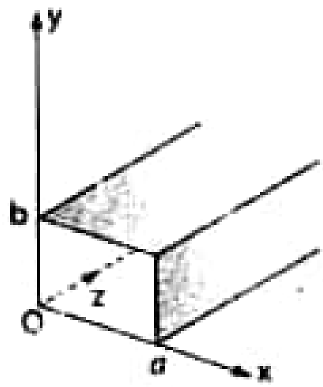

Fig 16.7-1: A rectangular waveguide.

From Eq. (16.6-5) we can write

$$k_c \cdot a = n\pi \quad \text{or} \quad \lambda_c = \frac{2a}{n}. \tag{16.6-11}$$

Physical importance of λ_g and λ_c can be derived as follows. Suppose for a given n and a, λ_0 is increased gradually. Equation (16.6-10) indicates that if λ_0 becomes greater than λ_c then λ_g becomes imaginary. In this case the exponent $jk_g z$ in Eq. (16.6-6) becomes negative for $z > 0$, and the wave becomes exponentially damped in z, instead of propagating. Thus, for the waves to propagate between the plates we must have

$$\lambda_0 < \lambda_c = \frac{2a}{n}.$$

For this λ_c is called *cut off wavelength*. It is the longest wavelength that can propagate for a given mode (n value).

λ_g is known as *guide wavelength*. From Eq. (16.6-10),

$$\lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}. \tag{16.6-12}$$

Obviously λ_g is greater than corresponding free space wavelength.

Equation (16.6-6) indicates that the wave propagates along z-axis with an apparent phase velocity

$$v_p = \frac{\omega}{k_g} = \frac{\omega}{k \sin\theta} = \frac{c}{\sin\theta}. \tag{16.6-13}$$

As $\sin\theta \le 1$, $v_p \ge c$. For $\theta = 0$, i.e., $\lambda_0 = \lambda_c$, v_p becomes infinite. This result appears to contradict the principle of relativity. In reality it does not violate the principle of relativity because signals are always transmitted with group velocity and not with phase velocity. Phase velocity is not the velocity of propagation of any physical quantity. It simply represents the velocity of a point of constant phase on the wave. The energy propagates down the guide with group velocity, which is always smaller than c.

The group velocity v_g is given by

$$v_g = \frac{d\omega}{dk_g}.$$

From Eqs. (16.6-7) and (16.6-11) we can write

$$k_g^2 = k^2 \sin^2\theta = k^2(1 - \cos^2\theta) = k^2 - k_c^2 = k^2 - \frac{n^2\pi^2}{a^2}$$

Therefore,

$$k_g^2 = \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2} \quad \left[\because c = \frac{\omega}{k}\right] \tag{16.6-14}$$